

Slavnov-Taylor identities for primordial perturbations

Lasha Berezhiani and Justin Khoury

Center for Particle Cosmology, Department of Physics & Astronomy,
University of Pennsylvania, 209 South 33rd Street, Philadelphia, PA 19104 U.S.A.

E-mail: lashaber@gmail.com, jkhoury@sas.upenn.edu

Received September 24, 2013

Accepted December 25, 2013

Published February 4, 2014

Abstract. Correlation functions of adiabatic modes in cosmology are constrained by an infinite number of consistency relations, which relate $N + 1$ -point correlation functions with a soft-momentum scalar or tensor mode to a symmetry transformation on N -point correlation functions of hard-momentum modes. They constrain, at each order n , the q^n behavior of the soft limits. In this paper we show that all consistency relations derive from a single, master identity, which follows from the Slavnov-Taylor identity for spatial diffeomorphisms. This master identity is valid at any value of q and therefore goes beyond the soft limit. By differentiating it n times with respect to the soft momentum, we recover the consistency relations at each q order. Our approach underscores the role of spatial diffeomorphism invariance at the root of cosmological consistency relations. It also offers new insights on the necessary conditions for their validity: a physical contribution to the vertex functional must satisfy certain analyticity properties in the soft limit in order for the consistency relations to hold. For standard inflationary models, this is equivalent to requiring that mode functions have constant growing-mode solutions. For more exotic models in which modes do not “freeze” in the usual sense, the analyticity requirement offers an unambiguous criterion.

Keywords: inflation, cosmological perturbation theory

ArXiv ePrint: [1309.4461](https://arxiv.org/abs/1309.4461)



Contents

1	Introduction	1
2	Ward identities for electrodynamics	4
3	Slavnov-Taylor identities for cosmology	7
4	Consistency relations with two hard scalar modes	9
4.1	Analyticity assumption	11
4.2	Recovering the order q^0 and q consistency relations	11
4.3	Higher-order consistency relations	12
5	Time-dependent path integral formalism	14
6	Conclusion	16
A	Projection operators for electrodynamics	16
B	Gauge-fixing term for gravity	17
C	Properties of projectors	18

1 Introduction

In recent years there has been interest in consistency relations for primordial perturbations. The simplest one [1–9] relates an N -point correlation function with a soft scalar operator ζ to a scale transformation of the $N - 1$ -point correlation function without the soft mode:¹

$$\lim_{\vec{q} \rightarrow 0} \frac{1}{P_\zeta(q)} \langle \zeta(\vec{q}) \mathcal{O}(\vec{p}_1, \dots, \vec{p}_N) \rangle'_c = - \left(3(N-1) + \sum_{a=1}^N \vec{p}_a \cdot \frac{\partial}{\partial \vec{p}_a} \right) \langle \mathcal{O}(\vec{p}_1, \dots, \vec{p}_N) \rangle'_c. \quad (1.1)$$

Here, $\mathcal{O}(\vec{p}_1, \dots, \vec{p}_N)$ is an arbitrary equal-time product of scalar ζ and tensor γ_{ij} modes, with momenta $\vec{p}_1, \dots, \vec{p}_N$. There is an analogous relation involving a soft tensor $\gamma_{ij}(\vec{q})$ related to an anisotropic rescaling of the lower-point function [1]. The power of these relations lie in their generality: *Any* early universe scenario involving a single scalar degree of freedom (or single ‘clock’), and whose perturbations become constant at late times, must satisfy (1.1). Conversely, they can be violated if multiple fields contribute to density perturbations and/or ζ grows outside the horizon [12–16].²

¹Here ζ is the curvature perturbation in uniform-density gauge [10, 11], P_ζ is the power spectrum, $\langle \dots \rangle'$ are correlators without the momentum-conserving δ -function, and the subscript c denotes the connected part. See the main text for further details.

²The consistency relation (1.1) can be interpreted as the statement that primordial correlation functions in a suitable coordinate system vanish in the soft limit [17, 18]. Nevertheless, this statement is not tautological — the fact that the consistency relations can be violated implies that they constitute physical, observationally-testable probes of early universe physics.

The consistency relation (1.1) is a consequence of symmetry: it follows from the Ward identity for spontaneously broken spatial dilations [1, 7–9].³ More generally, scalar perturbations on *any* spatially-flat cosmological background non-linearly realize the full conformal group $SO(4, 1)$ on \mathbb{R}^3 [5, 6]. The origin of conformal symmetry is most easily seen in comoving gauge, where the spatial metric (ignoring tensors) $h_{ij} = a^2(t)e^{2\zeta(\vec{x}, t)}\delta_{ij}$ is conformally flat and hence invariant under the 10 conformal transformations on \mathbb{R}^3 . The symmetry breaking pattern is $SO(4, 1) \rightarrow$ spatial rotations + translations, with ζ playing the role of the Goldstone boson (or dilation) for the broken dilation and special conformal transformations (SCTs). The Ward identities associated with the SCTs also give rise to consistency relations [5]. These relate the order q behavior of an $N + 1$ -point correlation function with a soft ζ mode to a SCT on the N -point function.

Recently, it has been shown that cosmological perturbations enjoy an infinite number of non-linearly realized global symmetries [9]. These are residual diffeomorphisms mapping field configurations which fall off at infinity into those which do not. Nevertheless, certain linear combinations of these transformations can be smoothly extended to physical configurations which fall off at infinity, and as such constitute adiabatic modes [21]. These symmetries can be labeled by an integer $n \geq 0$ and involve both scalar and tensor perturbations.

The corresponding Ward identities imply *an infinite number of consistency relations* [9], of which (1.1) is the simplest case. At each order, they constrain — completely for $n = 0, 1$, and partially for $n \geq 2$ — the q^n behavior of an $N + 1$ -point correlation function with a soft scalar or tensor mode to a symmetry transformation on an N -point function. Schematically, they are of the form

$$\lim_{\vec{q} \rightarrow 0} \frac{\partial^n}{\partial q^n} \left(\frac{1}{P_\zeta(q)} \langle \zeta(\vec{q}) \mathcal{O} \rangle'_c + \frac{1}{P_\gamma(q)} \langle \gamma(\vec{q}) \mathcal{O} \rangle'_c \right) \sim - \frac{\partial^n}{\partial p^n} \langle \mathcal{O} \rangle'_c. \quad (1.2)$$

There are 3 independent relations for $n = 0$ (the dilation identity (1.1) involving a soft scalar, and two involving a soft tensor), 7 relations for $n = 1$ (the 3 SCT identities involving a soft scalar, and 4 involving a soft tensor), and 6 for each $n \geq 2$ (4 involving a soft tensor, and 2 involving mixtures of soft scalar and tensor). The $n = 0$ and $n = 1$ relations were known from background-wave arguments. The $n \geq 2$ relations were discovered in [9].

In this paper, we show that the consistency relations (1.2) all derive from a single, *master identity*, which follows from the Slavnov-Taylor identity for spatial diffeomorphisms. This master identity is valid at any value of q and therefore goes beyond the soft limit. By differentiating it n times with respect to q and setting $q = 0$, we recover (1.2) at each order. Our approach underscores the role of diffeomorphism invariance at the root of cosmological consistency relations. It also offers insights on the necessary assumptions for their validity.

We will derive the master identity in two independent ways: first, using the fixed-time path-integral approach introduced in [8]; second, using the 4d path integral. For simplicity, we focus here on soft 3-point functions, with the hard momenta given by scalar modes. The generalization to more general correlation functions should be straightforward. Let us illustrate the results in the fixed-time approach, for concreteness. The master identity takes the form

$$\frac{1}{3} q_i \Gamma^{\zeta\zeta\zeta}(\vec{q}, \vec{p}, -\vec{q} - \vec{p}) + 2q^j \Gamma_{ij}^{\gamma\zeta\zeta}(\vec{q}, \vec{p}, -\vec{q} - \vec{p}) = q_i \Gamma_\zeta(p) - p_i \left(\Gamma_\zeta(|\vec{q} + \vec{p}|) - \Gamma_\zeta(p) \right), \quad (1.3)$$

³See [19, 20] for recent derivations of (1.1) using holographic arguments.

where $\Gamma^{\zeta\zeta\zeta}$ and $\Gamma_{ij}^{\gamma\zeta\zeta}$ are respectively the cubic vertex functions for 3 scalars, and for 2 scalars–1 tensor, each without the momentum-conserving delta function, while Γ_ζ is the inverse scalar propagator.⁴ The solution for $\frac{1}{3}\delta_{ij}\Gamma^{\zeta\zeta\zeta}(\vec{q}, \vec{p}, -\vec{q}-\vec{p}) + 2\Gamma_{ij}^{\gamma\zeta\zeta}(\vec{q}, \vec{p}, -\vec{q}-\vec{p})$ can be obtained as a power series around $q = 0$, up to an arbitrary symmetric, transverse matrix A_{ij} . This arbitrary term is model-dependent, and hence contains physical information about the underlying theory. It stems from the fact that (1.3) only constrains the longitudinal components of the vertex functions. *The key assumption underlying the consistency relations is that A_{ij} is analytic in q , specifically that it starts at $\mathcal{O}(q^2)$.* For standard inflationary scenarios, we will see that this is equivalent to the usual assumption of constant asymptotic solutions for the mode functions. For more exotic examples, such as khronon inflation [22], our criterion is the unambiguous one.

Up to q^2 order, therefore, we can isolate $\Gamma^{\zeta\zeta\zeta}$, and then convert to correlation functions as usual: $\langle \zeta_{\vec{p}_1} \zeta_{\vec{p}_2} \zeta_{\vec{p}_3} \rangle = P_\zeta(p_1) P_\zeta(p_2) P_\zeta(p_3) \Gamma^{\zeta\zeta\zeta}(\vec{p}_1, \vec{p}_2, \vec{p}_3)$. The result is

$$\begin{aligned} \frac{\langle \zeta_{\vec{q}} \zeta_{\vec{p}} \zeta_{-\vec{p}-\vec{q}} \rangle'}{P_\zeta(q)} &= - \left(3 + \vec{p} \cdot \frac{\partial}{\partial \vec{p}} \right) P_\zeta(p) \\ &\quad - \frac{1}{2} q^i \left(6 \frac{\partial}{\partial p^i} - p_i \frac{\partial^2}{\partial p^j \partial p_j} + 2 p_j \frac{\partial^2}{\partial p^j \partial p^i} \right) P_\zeta(p) + \mathcal{O}(q^2). \end{aligned} \quad (1.4)$$

The first line is the dilation consistency relation and agrees with (1.1) for the case of interest. The second line reproduces the SCT consistency relation [5]. At order q^2 and higher, the physical term A_{ij} contributes. However, its contribution can be removed order by order by taking linear combinations of $\langle \zeta\zeta\zeta \rangle$ and $\langle \gamma\zeta\zeta \rangle$, and taking a suitable projection. In this way, we will recover the general consistency relations (1.2).

Interestingly, the essence of our method is fully captured by Quantum Electrodynamics (QED), which we present along the lines of [23] in section 2. The idea is simple: although the gauge is usually fixed by a gauge-fixing term, the gauge symmetry does give us an information about the interaction vertices. In other words, the gauge invariant part of the interaction vertices is constrained by the symmetry and the corrections coming from the gauge-fixing term can be accounted for explicitly. The resulting Ward-Takahashi identity [24, 25], as is well known, relates the (longitudinal part of the) photon-fermion vertex to the fermion propagator. By expanding this identity as a power series in the soft photon momentum q , we will obtain QED consistency relations analogous to ones obtain in [9] for cosmology.

The approach described in this paper is quite general and can be applied beyond the early universe, for instance, to derive consistency relations for the large scale structure [26–29]. It can also be applied to derive consistency relations for modified initial states [30–38], as well as consistency relations for multiple soft limits [39]. Another interesting arena is the conformal mechanism [40–46], whose consistency relations have been derived recently [47].

The paper is organized as follows. In section 2, we begin with the warm-up example of QED and outline all of its relevant properties. In section 3, we turn to the derivation of the Slavnov-Taylor identity for cosmological perturbations, first using the fixed-time/three-dimensional Euclidian path-integral method proposed in [8]. In section 4, we illustrate how the consistency relations derive from the Slavnov-Taylor identity, focusing for simplicity on consistency relations involving two hard scalar modes with a soft scalar or tensor mode. In section 5, we rederive these results, this time using the conventional four-dimensional in-in path integral. Some of technical details of our derivations have been relegated to a series of

⁴Note that $\Gamma_{ij}^{\gamma\zeta\zeta}$ is traceless ($\delta^{ij}\Gamma_{ij}^{\gamma\zeta\zeta} = 0$), but not necessarily transverse ($q^j\Gamma_{ij}^{\gamma\zeta\zeta} \neq 0$).

appendices. We summarize the results in section 6 discuss further applications of the method outlined here.

2 Ward identities for electrodynamics

As an abelian warm-up to the cosmological case, consider the Ward-Takahashi identities for QED [24, 25], derived as a consequence of gauge symmetry [23] instead of its global subgroup. The generating functional for QED with a single Dirac fermion ψ is given by the following path integral

$$Z[J_\mu, \eta, \bar{\eta}] = \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{iS + iS_{\text{g.-f.}} + iS_{\text{ext.}}}, \quad (2.1)$$

where $S[A_\mu, \bar{\psi}, \psi]$ is the gauge invariant QED action, and

$$S_{\text{g.-f.}}[A_\mu] = -\frac{1}{2\xi} \int d^4x (\partial^\mu A_\mu)^2; \quad S_{\text{ext.}}[A_\mu, \bar{\psi}, \psi] = \int d^4x (J^\mu A_\mu + \bar{\eta}\psi + \bar{\psi}\eta) \quad (2.2)$$

are the gauge-fixing term and external current contributions, respectively.⁵

To derive the Ward-Takahashi identities, we perform an infinitesimal gauge transformation⁶

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda; \quad \psi \rightarrow \psi - i\Lambda\psi, \quad (2.3)$$

where $\Lambda(x)$ is an infinitesimal gauge parameter. Since S and the integration measure are both gauge invariant, the variation of the generating functional is

$$\begin{aligned} \delta Z[J_\mu, \eta, \bar{\eta}] &= i \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{iS + iS_{\text{g.-f.}} + iS_{\text{ext.}}} \int d^4x \Lambda(x) \left[-\frac{\square}{\xi} \partial^\mu A_\mu - \partial^\mu J_\mu - i\bar{\eta}\psi + i\bar{\psi}\eta \right] \\ &= i \int d^4x \Lambda(x) \left[\frac{i\square}{\xi} \partial^\mu \frac{\delta}{\delta J^\mu} - \partial^\mu J_\mu - \bar{\eta} \frac{\delta}{\delta \bar{\eta}} + \eta \frac{\delta}{\delta \eta} \right] Z[J, \bar{\eta}, \eta]. \end{aligned} \quad (2.4)$$

The generating functional should be invariant ($\delta Z = 0$) under (2.3), since it is merely a field redefinition. Since $\Lambda(x)$ is arbitrary, this leads to the functional differential equation

$$\left[\frac{i\square}{\xi} \partial^\mu \frac{\delta}{\delta J^\mu} - \partial^\mu J_\mu - \bar{\eta} \frac{\delta}{\delta \bar{\eta}} + \eta \frac{\delta}{\delta \eta} \right] Z[J, \bar{\eta}, \eta] = 0. \quad (2.5)$$

Clearly, the generating functional of connected diagrams, $W \equiv -i \ln Z$, obeys a similar differential equation. Performing the standard Legendre transform to the vertex functional $\Gamma \equiv W - S_{\text{ext.}}$, and using the standard relations $J^\mu = -\frac{\delta \Gamma}{\delta A_\mu}$, $A_\mu = \frac{\delta W}{\delta J^\mu}$, etc., this implies

$$-\frac{\square}{\xi} \partial^\mu A_\mu + \partial_\mu \frac{\delta \Gamma}{\delta A_\mu} + i\psi \frac{\delta \Gamma}{\delta \psi} - i\bar{\psi} \frac{\delta \Gamma}{\delta \bar{\psi}} = 0. \quad (2.6)$$

Note that nowhere did we need the explicit form of the QED action S — all we used was its invariance under (2.3). Therefore, the identity (2.6) holds more generally for *any* gauge invariant action. We will henceforth assume this most general situation.

⁵Here, ξ is the standard gauge parameter, e.g. $\xi = 1$ is Feynman gauge.

⁶The fermion is assumed to carry a unit charge.

By varying (2.6) a number of times with respect to the fields, and setting the fields to zero after variation, one can obtain various relations among the vertices of the theory. For instance, varying with respect to ψ and $\bar{\psi}$ gives (in momentum space):

$$q^\mu \Gamma_\mu^{A\bar{\psi}\psi}(q, p, -p - q) = \Gamma_\psi(p + q) - \Gamma_\psi(p), \quad (2.7)$$

where $\Gamma_\mu^{A\bar{\psi}\psi} = \frac{\delta^3 \Gamma}{\delta A^\mu \delta \psi \delta \bar{\psi}}$ is the three-point vertex, and $\Gamma_\psi(p) = \frac{\delta^2 \Gamma}{\delta \psi \delta \bar{\psi}}$ is the inverse fermion propagator. Equation (2.7) is the celebrated Ward-Takahashi identity [24, 25]. It exhibits the constraint that must be obeyed by the vertex functionals.

We are interested in deriving the identity for correlation functions, which are related to the vertex functionals as follows:

$$\begin{aligned} P_\psi(p) &\equiv \langle \bar{\psi}_p \psi_{-p} \rangle' = 1/\Gamma_\psi(p), \\ \langle A_q^\mu \bar{\psi}_p \psi_{-p-q} \rangle' &= -\langle A_q^\mu A_{-q}^\nu \rangle' P_\psi(p) P_\psi(p + q) \Gamma_\nu^{A\bar{\psi}\psi}(q, p, -p - q), \end{aligned} \quad (2.8)$$

where primes indicate correlators with the delta function removed:

$$\langle \mathcal{O}(k_1, \dots, k_N) \rangle \equiv (2\pi)^4 \delta^4(k_1 + \dots + k_N) \langle \mathcal{O}(k_1, \dots, k_N) \rangle'. \quad (2.9)$$

This makes clear that the quantity we would like to solve for is $P_\psi(p) P_\psi(p + q) \Gamma_\nu^{A\bar{\psi}\psi}(q, p, -p - q)$. Rewriting (2.7) in terms of this quantity, we obtain

$$q^\mu P_\psi(p) P_\psi(p + q) \Gamma_\mu^{A\bar{\psi}\psi}(q, p, -p - q) = P_\psi(p) - P_\psi(p + q). \quad (2.10)$$

It is straightforward to see that the most general solution to this equation is given by

$$P_\psi(p) P_\psi(p + q) \Gamma_\mu^{A\bar{\psi}\psi}(q, p, -p - q) = -\frac{\partial P_\psi(p)}{\partial p^\mu} - \sum_{n=1}^{\infty} \frac{q^{\alpha_1} \dots q^{\alpha_n}}{(n+1)!} \frac{\partial^{n+1} P_\psi(p)}{\partial p^\mu \partial p^{\alpha_1} \dots \partial p^{\alpha_n}} + C_\mu, \quad (2.11)$$

where C_μ is an arbitrary transverse vector,

$$q^\mu C_\mu = 0. \quad (2.12)$$

Its more general form is therefore

$$C_\mu = (q^2 \eta_{\mu\alpha} - q_\mu q_\alpha) v^\alpha(q, p) + q^\alpha M_{\mu\alpha}(q, p), \quad (2.13)$$

where v^μ is an arbitrary vector, and $M_{\mu\nu}$ is anti-symmetric. The vector C_μ represents the part of the cubic vertex which is not fixed by symmetry arguments only, but depends on the details of the theory. It therefore encodes physical information about the theory.

Now comes the key assumption: if the theory is *local*, which is of course the case for QED, then C_μ *should be analytic in q* . In terms of the general decomposition (2.13), this implies that both v^μ and $M_{\nu\mu}$ start at order q^0 . In particular, it follows that C_μ does not contribute at q^0 order, thus the cubic vertex is determined by the derivative of the inverse fermion propagator at leading order:

$$\Gamma_\mu^{A\bar{\psi}\psi}(0, p, -p) = \frac{\partial \Gamma_\psi(p)}{\partial p^\mu}. \quad (2.14)$$

This is the QED analogue of Maldacena's consistency relation (1.1).⁷

⁷Notice that (2.14) is satisfied not only for fermion QED, but for scalar QED as well. This traces back to the fact that fermions and scalars have identical transformation properties under the gauge symmetry.

At the next order in q , however, C_μ can contribute through the $M_{\mu\nu}$ term. For instance, with fermions it can take the form $C^\mu = q_\nu[\gamma^\nu, \gamma^\mu]$, where the γ 's are the usual Dirac matrices. This will arise if the theory includes an anomalous magnetic dipole interaction, $F_{\mu\nu}\bar{\psi}\gamma^\mu\gamma^\nu\psi$. More generally, we see that C_μ encodes information about non-minimal photon couplings to the fermions. It vanishes identically for QED, where the photon-fermion coupling is minimal.

We can translate the identity (2.11) to a statement about correlation functions by contracting the vertex functional with the appropriate Green's functions. Specializing to Lorentz gauge, $\partial^\mu A_\mu = 0$, we have

$$\frac{\langle A_q^\mu \bar{\psi}_p \psi_{-p-q} \rangle'}{P_A(q)} = -P^{\mu\nu}(\hat{q}) P_\psi(p) P_\psi(p+q) \Gamma_\nu^{A\bar{\psi}\psi}(q, p, -p-q), \quad (2.15)$$

where P_A is defined as $\langle A_q^\mu A_{-q}^\nu \rangle' = P_A(q) P^{\mu\nu}(\hat{q})$, with $P_{\mu\nu}(\hat{q}) = \eta_{\mu\nu} - \hat{q}_\mu \hat{q}_\nu$, $\hat{q}^\mu \equiv q^\mu/q$, denoting the transverse projector.

Ideally we would like to derive model-independent (i.e., C_μ -independent) relations among the correlation functions. To do so, we must apply suitable component operators $P_{\mu\ell_1\ldots\ell_n\nu m_1\ldots m_n}$ on each term of the Taylor series (2.11) such that C_μ is projected out at each order. In other words, the C_μ contribution on the right-hand side of

$$\begin{aligned} & \lim_{q \rightarrow 0} P_{\mu\ell_1\ldots\ell_n\nu m_1\ldots m_n}(\hat{q}) \frac{\partial^n}{\partial q_{m_1} \cdots \partial q_{m_n}} \left[\frac{\langle A_q^\nu \bar{\psi}_p \psi_{-p-q} \rangle'}{P_A(q)} \right] \\ &= - \lim_{q \rightarrow 0} P_{\mu\ell_1\ldots\ell_n\nu m_1\ldots m_n}(\hat{q}) \frac{\partial^n}{\partial q_{m_1} \cdots \partial q_{m_n}} \left(P^{\nu\alpha}(\hat{q}) P_\psi(p) P_\psi(p+q) \Gamma_\alpha^{A\bar{\psi}\psi}(q, p, -p-q) \right) \end{aligned} \quad (2.16)$$

should drop out. We claim this is achieved if $P_{\mu\ell_1\ldots\ell_n\nu m_1\ldots m_n}$ is:

1. Symmetric in the $(\mu, \ell_1, \ldots, \ell_n)$ indices and in the $(\nu, m_1 \ldots m_n)$ indices.
2. Symmetric under the interchange of sets of indices: $(\mu, \ell_1, \ldots, \ell_n) \leftrightarrow (\nu, m_1 \ldots m_n)$.
3. Traceless:

$$P_{\mu\ell_2\ldots\ell_n\nu m_1\ldots m_n}^\mu(\hat{q}) = 0. \quad (2.17)$$

4. Transverse with respect to q :

$$q^\mu P_{\mu\ell_1\ldots\ell_n\nu m_1\ldots m_n}(\hat{q}) = 0. \quad (2.18)$$

In appendix A, we show explicitly that C_μ is indeed successfully projected out by the projector defined above.⁸

⁸The properties of this projector can be further motivated through an ‘‘adiabaticity’’ argument similar to that given in [9] for the cosmological case. The Lorentz gauge condition $\partial^\mu A_\mu = 0$ is preserved by gauge transformations $\delta A_\mu = \partial_\mu \lambda$, where λ is a harmonic function: $\square \lambda = 0$. Expanding as a Taylor series about the origin, we have $\lambda(x) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} M_{\ell_0\ldots\ell_n} x^{\ell_0} \cdots x^{\ell_n}$, where the array $M_{\ell_0\ldots\ell_n}$ is fully symmetric and traceless. (We have ignored the constant term in the expansion, since it leaves A_μ invariant.) In momentum space, this generates

$$\delta A_\mu(q) = \frac{(-i)^n}{n!} M_{\mu\ell_1\ldots\ell_n} \frac{\partial^n}{\partial q_{\ell_1} \cdots \partial q_{\ell_n}} (2\pi)^4 \delta^4(q). \quad (2.19)$$

For this configuration to be extendible to a *physical* mode, with suitable fall-off behavior at spatial infinity, we imagine smoothing out the momentum profile around $q = 0$. To ensure that transversality is preserved in Fourier space at finite momentum, $q^\mu \delta A_\mu = 0$, we must let the $M_{\mu\ell_1\ldots\ell_n}$ coefficients become \hat{q} -dependent such that

$$q^\mu M_{\mu\ell_1\ldots\ell_n}(\hat{q}) = 0. \quad (2.20)$$

In other words, $M_{\mu\ell_1\ldots\ell_n}$ is fully symmetric, traceless and transverse. The corresponding projector $P_{\mu\ell_1\ldots\ell_n\nu m_1\ldots m_n}(\hat{q})$ appearing in the identities must therefore satisfy the properties listed in the main text.

Making use of this fact when substituting (2.11) into (2.16), we obtain the following consistency relations

$$\lim_{q \rightarrow 0} P_{\mu\ell_1 \dots \ell_n \nu m_1 \dots m_n}(\hat{q}) \frac{\partial^n}{\partial q_{m_1} \dots \partial q_{m_n}} \left[\frac{\langle A_q^\nu \bar{\psi}_p \psi_{-p-q} \rangle'}{P_A(q)} \right] = - \frac{P_{\mu\ell_1 \dots \ell_n \nu m_1 \dots m_n}(\hat{q})}{n+1} \frac{\partial^{n+1} P_\psi}{\partial p_\nu \partial p_{m_1} \dots \partial p_{m_n}}.$$

These are very similar in form to the Ward identities for cosmological perturbations derived in [9]. In the following sections we will reproduce these identities as a consequence of spatial diffeomorphism invariance. The derivation is closely analogous to the one above, with the replacement of U(1) gauge symmetry by diffeomorphism invariance. Because of the non-Abelian nature of the latter we will refer to the resulting identities similar as Slavnov-Taylor identities for cosmology.

3 Slavnov-Taylor identities for cosmology

We now turn to the derivation of cosmological consistency relations. Our method follows Slavnov's classic work [23], applied to cosmology.⁹ The non-abelian nature of the symmetries of interest (namely, the diffeomorphism invariance of GR) complicates the derivation to some extent. In particular, the gauge-fixing term, which in the abelian case dropped out of the identity (2.7), does contribute to the Slavnov-Taylor identities in the non-abelian case. As shown in appendix B, however, the gauge-fixing term only contributes at loop order. While this contribution can be accounted for explicitly if desired, we avoid the unnecessary complications and work at tree-level.

For simplicity, we begin the demonstration of our method in the framework of the fixed-time path-integral formalism of [8]. The basic idea is simple: since we are solely interested in correlation functions of fields evaluated at the final time (as opposed to unequal-time correlators, or correlators involving time-derivatives of the fields), it is convenient to work with a three-dimensional Euclidean path integral over field configurations at the final time, with the “history” information being encoded in the wavefunction. The fixed-time formalism makes the derivation simpler and more transparent. In section 5, we will reproduce the same results using the four-dimensional in-in path integral.

We consider the diffeomorphism invariant theory of the metric degrees of freedom $g_{\mu\nu}$ and inflaton ϕ , around the spatially-flat Friedmann-Robertson-Walker background

$$\bar{g}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) d\vec{x}^2; \quad \phi = \bar{\phi}(t), \quad (3.1)$$

and parameterize the excitations as

$$g_{\mu\nu} = \bar{g}_{\mu\nu}(t) + a^2(t) h_{\mu\nu}; \quad \phi = \bar{\phi}(t) + \varphi. \quad (3.2)$$

According to [8], the correlation functions at fixed time t can be conveniently described by the Euclidean generating functional

$$\begin{aligned} Z[T, J] &= \int \mathcal{D}h_{ij} \mathcal{D}\varphi |\Psi[h, \varphi, t]|^2 e^{S_{\text{ext.}}}; \\ S_{\text{ext.}} &= \int d^3x (h_{ij} T^{ij} + \varphi J), \end{aligned} \quad (3.3)$$

⁹For the flat space considerations, see [48, 49].

where T_{ij} and J represent tensor and scalar currents, respectively, while $\Psi[h, \varphi, t]$ is the wavefunctional at time t .¹⁰ Note that the auxiliary fields h_{00} and h_{0i} (equivalently, the lapse function and shift vector) have been integrated out using the constraint equations [1], so that the path integral is over the spatial metric h_{ij} only.

Since time is fixed in this approach, the time re-parametrization symmetry is explicitly broken by the formalism. The symmetries at hand are spatial diffeomorphisms $x^i \rightarrow x^i - \xi^i$, under which the fields transform as

$$\begin{aligned} h_{ij} &\rightarrow h_{ij} + \partial_i \xi_j + \partial_j \xi_i + \xi^k \partial_k h_{ij} + h_{ik} \partial_j \xi^k + h_{jk} \partial_i \xi^k; \\ \varphi &\rightarrow \varphi + \partial_k \varphi \xi^k. \end{aligned} \quad (3.4)$$

From now on, all the indices are assumed to be raised and lowered using δ_{ij} . Analogously to the QED case — see (2.4) —, the invariance of the generating functional under this field redefinition leads to

$$\begin{aligned} 0 &= \int \mathcal{D}h_{ij} \mathcal{D}\varphi |\Psi[h, \varphi, t]|^2 e^{S_{\text{ext}}} \int d^3x \xi^k \left\{ (\text{G.F.})_k - 2\partial_j T_k^j + \partial_k h_{ij} T^{ij} - 2\partial_j (h_{ik} T^{ij}) + \partial_k \varphi J \right\} \\ &= \int d^3x \xi^k \left\{ (\text{G.F.})_k - 2\partial_j T_k^j + \partial_k \left(\frac{\delta}{\delta T^{ij}} \right) T^{ij} - 2\partial_j \left(\frac{\delta}{\delta T^{ik}} T^{ij} \right) + \partial_k \left(\frac{\delta}{\delta J} \right) J \right\} Z[T, J], \end{aligned} \quad (3.5)$$

where in the last step we have made the replacements $\varphi \rightarrow \frac{\delta}{\delta J}$ and $h_{ij} \rightarrow \frac{\delta}{\delta T^{ij}}$. Here, $(\text{G.F.})_k$ denotes terms arising from the variation of the gauge-fixing term; we will be schematic about it until its explicit form becomes important. Since ξ^k is arbitrary, the integrand itself must vanish. Rewriting the result in terms of $W = \ln Z$, the generator of connected amplitudes, we obtain

$$(\text{G.F.})_k - 2\partial_j T_k^j(x) + \partial_k \left(\frac{\delta W}{\delta T^{ij}(x)} \right) T^{ij}(x) - 2\partial_j \left(\frac{\delta W}{\delta T^{ik}(x)} T^{ij}(x) \right) + \partial_k \left(\frac{\delta W}{\delta J(x)} \right) J(x) = 0. \quad (3.6)$$

We can convert (3.6) to an equation for the vertex functional by means of the Legendre transform

$$\Gamma[h, \varphi] = W[T, J] - \int d^3x (h_{ij} T^{ij} + \varphi J), \quad (3.7)$$

which implies

$$\frac{\delta \Gamma}{\delta h_{ij}} = -T^{ij}; \quad \frac{\delta W}{\delta T^{ij}} = h_{ij}; \quad (3.8)$$

$$\frac{\delta \Gamma}{\delta \varphi} = -J; \quad \frac{\delta W}{\delta J} = \varphi. \quad (3.9)$$

The resulting equation for Γ is

$$(\text{G.F.})_k + 2\partial_j \frac{\delta \Gamma}{\delta h_{jk}} = (\partial_k h_{ij}) \frac{\delta \Gamma}{\delta h_{ij}} - 2\partial_j \left(h_{ik} \frac{\delta \Gamma}{\delta h_{ij}} \right) + \partial_k \varphi \frac{\delta \Gamma}{\delta \varphi}. \quad (3.10)$$

At this point, we specialize to comoving gauge (or ‘ ζ gauge’), defined by

$$\varphi = 0; \quad H_{ij} \equiv \delta_{ij} + h_{ij} = e^{2\zeta} \hat{h}_{ij}, \quad \text{with} \quad \det \hat{h} = 1. \quad (3.11)$$

¹⁰The peculiarities related to the gauge-fixing term will be addressed separately and can be ignored for the moment.

It follows that

$$\begin{aligned}\zeta &= \frac{1}{6} \ln \det H; \\ \hat{h}_{ij} &= \frac{H_{ij}}{(\det H)^{1/3}}.\end{aligned}\tag{3.12}$$

The variational derivative can be converted to the new variables using

$$\begin{aligned}\frac{\delta\Gamma}{\delta h_{ij}} &= \frac{\delta\Gamma}{\delta\zeta} \frac{\delta\zeta(h)}{\delta h_{ij}} + \frac{\delta\Gamma}{\delta\hat{h}_{k\ell}} \frac{\delta\hat{h}_{k\ell}(h)}{\delta h_{ij}} \\ &= e^{-2\zeta} \left\{ \frac{1}{6} [\hat{h}^{-1}]_{ij} \frac{\delta\Gamma}{\delta\zeta} + \left[\delta_{ik} \delta_{j\ell} - \frac{1}{3} \hat{h}_{k\ell} [\hat{h}^{-1}]_{ij} \right] \frac{\delta\Gamma}{\delta\hat{h}_{k\ell}} \right\} \\ &= e^{-2\zeta} \left\{ \frac{1}{6} \delta_{ij} \frac{\delta\Gamma}{\delta\zeta} + \frac{\delta\Gamma}{\delta\gamma_{ij}} \right\} + \dots\end{aligned}\tag{3.13}$$

where in the last step we have introduced

$$\gamma_{ij} \equiv \ln \hat{h}_{ij}; \quad \gamma^i_i = 0.\tag{3.14}$$

Here and henceforth, the ellipses indicate terms that are higher-order in γ . As we will see shortly, these will not contribute at tree-level to the consistency relations of interest in this paper.

Substituting these results into (3.10), the Slavnov-Taylor identity reduces to

$$\frac{1}{\alpha} \left(\vec{\nabla}^2 \partial_j \gamma_{ij} + \partial_i \partial_j \partial_k \gamma_{jk} \right) + 2 \partial_j \left(\frac{1}{6} \delta_{ij} \frac{\delta\Gamma}{\delta\zeta} + \frac{\delta\Gamma}{\delta\gamma_{ij}} \right) = \partial_i \zeta \frac{\delta\Gamma}{\delta\zeta} + \dots\tag{3.15}$$

Note that we have only included the tree-level contribution of the gauge-fixing term explicitly, with α denoting a gauge-fixing parameter. Hence this identity (and the consistency relations that derive from it) only holds at tree-level. This is one of the key results of this paper. By varying this identity a number of times with respect to ζ and γ , it is straightforward to obtain various consistency relations among the vertices of the theory.

4 Consistency relations with two hard scalar modes

We illustrate how consistency relations derive from (3.15) in the simplest case of a soft-momentum ζ or γ mode coupled to two hard-momenta ζ modes. The generalization to higher-point correlation functions is straightforward.

The consistency relations with two hard-momenta scalar modes are obtained by varying (3.15) with respect to $\zeta(\vec{x}_1)$ and $\zeta(\vec{x}_2)$ and then setting $\zeta = \gamma_{ij} = 0$. Since the ellipses contain terms with powers of γ , they will all vanish upon setting $\gamma = 0$, as advocated. Expressing the result in Fourier space, we obtain¹¹

$$\frac{1}{3} q_i \Gamma^{\zeta\zeta\zeta}(\vec{q}, \vec{p}, -\vec{q} - \vec{p}) + 2 q^j \Gamma_{ij}^{\gamma\zeta\zeta}(\vec{q}, \vec{p}, -\vec{q} - \vec{p}) = q_i \Gamma_\zeta(p) - p_i \left(\Gamma_\zeta(|\vec{q} + \vec{p}|) - \Gamma_\zeta(p) \right).\tag{4.2}$$

¹¹The relevant relations are

$$\begin{aligned}\int d^3x_1 d^3x_2 e^{-i(\vec{p}_1 \cdot \vec{x}_1 + \vec{p}_2 \cdot \vec{x}_2)} \frac{\delta^2 \Gamma}{\delta\zeta(\vec{x}_1) \delta\zeta(\vec{x}_2)} \Big|_{\zeta=\gamma=0} &= (2\pi)^3 \delta^3(\vec{p}_1 + \vec{p}_2) \Gamma_\zeta(p_1); \\ \int d^3x_1 d^3x_2 d^3x_3 e^{-i \sum \vec{p}_i \cdot \vec{x}_i} \frac{\delta^3 \Gamma}{\delta\zeta(\vec{x}_1) \delta\zeta(\vec{x}_2) \delta\zeta(\vec{x}_3)} \Big|_{\zeta=\gamma=0} &= (2\pi)^3 \delta^3(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) \Gamma^{\zeta\zeta\zeta}(\vec{p}_1, \vec{p}_2, \vec{p}_3); \\ \int d^3x_1 d^3x_2 d^3x_3 e^{-i \sum \vec{p}_i \cdot \vec{x}_i} \frac{\delta^3 \Gamma}{\delta\gamma_{ij}(\vec{x}) \delta\zeta(\vec{y}) \delta\zeta(\vec{z})} \Big|_{\zeta=\gamma=0} &= (2\pi)^3 \delta^3(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) \Gamma_{ij}^{\gamma\zeta\zeta}(\vec{p}_1, \vec{p}_2, \vec{p}_3).\end{aligned}\tag{4.1}$$

This is the master identity. By converting it to correlation functions, we will see how it recovers all known consistency relations involving a soft mode and two hard scalar modes.

We can translate (4.2) to a statement about correlation functions using $\Gamma_\zeta(p) = -P_\zeta^{-1}(p)$, as well as¹²

$$\begin{aligned}\langle \zeta_{\vec{q}} \zeta_{\vec{p}} \zeta_{-\vec{q}-\vec{p}} \rangle' &= P_\zeta(q) P_\zeta(p) P_\zeta(|\vec{q} + \vec{p}|) \Gamma^{\zeta\zeta\zeta}(\vec{q}, \vec{p}, -\vec{q} - \vec{p}); \\ \langle \gamma_{\vec{q}}^{ij} \zeta_{\vec{p}} \zeta_{-\vec{q}-\vec{p}} \rangle' &= \hat{P}^{ijkl}(\hat{q}) P_\gamma(q) P_\zeta(p) P_\zeta(|\vec{q} + \vec{p}|) \Gamma_{kl}^{\gamma\zeta\zeta}(\vec{q}, \vec{p}, -\vec{q} - \vec{p}),\end{aligned}\quad (4.5)$$

where \hat{P}_{ijkl} is the transverse, traceless tensor appearing in the graviton propagator:

$$\hat{P}_{ijkl}(\hat{q}) = P_{ik}P_{jl} + P_{il}P_{jk} - P_{ij}P_{kl}, \quad (4.6)$$

with $P_{ij} \equiv \delta_{ij} - \hat{q}_i \hat{q}_j$ denoting the transverse projector.

Proceeding analogously to the QED case, we can solve (4.2) as a Taylor series around $q = 0$:

$$P_\zeta(p) P_\zeta(|\vec{q} + \vec{p}|) \left(\frac{1}{3} \delta_{ij} \Gamma^{\zeta\zeta\zeta}(\vec{q}, \vec{p}, -\vec{q} - \vec{p}) + 2 \Gamma_{ij}^{\gamma\zeta\zeta}(\vec{q}, \vec{p}, -\vec{q} - \vec{p}) \right) = K_{ij} + A_{ij}, \quad (4.7)$$

where

$$\begin{aligned}K_{ij} \equiv & -\delta_{ij} P_\zeta(p) - p_{(i} \frac{\partial P_\zeta(p)}{\partial p^{j)}} - \sum_{n=1}^{\infty} \frac{q_{\alpha_1} \dots q_{\alpha_n}}{n!} \left[\delta_{ij} \frac{\partial^n}{\partial p_{\alpha_1} \dots \partial p_{\alpha_n}} + \frac{p_i}{n+1} \frac{\partial^{n+1}}{\partial p_j \partial p_{\alpha_1} \dots \partial p_{\alpha_n}} \right. \\ & \left. + \frac{p_j}{n+1} \frac{\partial^{n+1}}{\partial p_i \partial p_{\alpha_1} \dots \partial p_{\alpha_n}} - \frac{p_{\alpha_1}}{n+1} \frac{\partial^{n+1}}{\partial p_i \partial p_j \partial p_{\alpha_2} \dots \partial p_{\alpha_n}} \right] P_\zeta(p),\end{aligned}\quad (4.8)$$

and A_{ij} is an arbitrary symmetric and transverse matrix:

$$q^j A_{ij} = 0. \quad (4.9)$$

Therefore it is of the general form¹³

$$A_{ij} = \epsilon_{ikm} \epsilon_{j\ell n} q^k q^\ell \left(a(\vec{p}, \vec{q}) \delta^{mn} + b(\vec{p}, \vec{q}) p^m p^n \right), \quad (4.10)$$

where a and b are *a priori* arbitrary scalar functions of the momenta. The ambiguous nature of (4.7) originates from the form of (4.2), which only constrains the longitudinal components of the quantities at hand. Note that A_{ij} is the analogue of C_μ for the QED case — see (2.11)

¹²Primed correlation functions are defined by removing the delta function:

$$\langle \mathcal{O}(\vec{q}, \vec{k}_1, \dots, \vec{k}_N) \rangle = (2\pi)^3 \delta^3(\vec{q} + \vec{k}_1 + \dots + \vec{k}_N) \langle \mathcal{O}(\vec{q}, \vec{k}_1, \dots, \vec{k}_N) \rangle'. \quad (4.3)$$

In particular, the power spectra are defined by

$$\begin{aligned}P_\zeta(k) &= \langle \zeta_{\vec{k}} \zeta_{\vec{k}'} \rangle'; \\ P_\gamma(k) &= \frac{1}{4} \delta_{ik} \delta_{j\ell} \langle \gamma_{\vec{k}}^{ij} \gamma_{\vec{k}'}^{\ell\ell'} \rangle'.\end{aligned}\quad (4.4)$$

¹³This can be seen by noting that the building blocks at our disposal are δ_{ij} , q_i and p_i . There is also the Levi-Cevita symbol ϵ_{ijk} , but by symmetry A_{ij} must be proportional to an even number of these, which is equivalent to products of δ 's.

and (2.12). This array encodes the physical part of the cubic vertices which are not fixed by symmetry arguments only, and hence depends on the details of the theory.

Isolating the trace and traceless parts of (4.7) allows us to solve for the individual vertices. Translating to correlation functions, we obtain

$$\begin{aligned}\frac{\langle \zeta_{\vec{q}} \zeta_{\vec{p}} \zeta_{-\vec{q}-\vec{p}} \rangle'}{P_\zeta(q)} &= K + A; \\ \frac{\langle \gamma_{\vec{q}}^{ij} \zeta_{\vec{p}} \zeta_{-\vec{q}-\vec{p}} \rangle'}{P_\gamma(q)} &= \frac{1}{2} \hat{P}^{ijk\ell}(\hat{q})(K_{k\ell} + A_{k\ell}).\end{aligned}\quad (4.11)$$

To derive consistency relations from these, we must make an important assumption about the behavior of the arbitrary array A_{ij} in the squeezed limit.

4.1 Analyticity assumption

The key assumption for the validity of the consistency relations, as in the QED case, is that the functions a and b are analytic in q , i.e., that the physical term A_{ij} starts at order q^2 . This locality assumption on the effective action¹⁴ (or vertex functional Γ) is a non-trivial one: although GR is local by construction, recall that we are working in the framework where the lapse function and shift vector have been integrated out, resulting in a spatially non-local action for ζ and γ .

In particular, let us see how this relates to the usual adiabaticity assumption, i.e., that the growing mode solutions are constant. Recall that non-local terms at cubic order arise from integrating out the shift vector, whose solution (at linear order) includes

$$N_i \supset -a^2 \frac{\dot{H}}{H^2} \frac{q_i}{q^2} \dot{\zeta}. \quad (4.12)$$

For the adiabatic mode, however, $\dot{\zeta} \propto q^2$, and this contribution becomes local.¹⁵ Conversely, in models where ζ is not constant outside the horizon (because of background instabilities [13]), the locality assumption is violated and the consistency relations will not hold. Similarly, this also explains why certain consistency relations fail in spatially non-local models, such as kronon inflation [22].

In the remainder of the section we will see how the consistency relations to all orders in q follow from (4.11), given the analyticity assumption.

4.2 Recovering the order q^0 and q consistency relations

Since the physical term A_{ij} kicks in at q^2 order, by assumption, the 3-point functions is uniquely determined by the 2-point function to zeroth and first order in q . The only contribution to the right-hand side of (4.11) at this order comes from K_{ij} , whose expansion is given by

$$K_{ij} = -\delta_{ij} P_\zeta(p) - p_{(i} \frac{\partial P_\zeta}{\partial p^{j)}} - \frac{1}{2} q_\ell \left[p_i \frac{\partial^2 P_\zeta(p)}{\partial p_j \partial p_\ell} + p_j \frac{\partial^2 P_\zeta(p)}{\partial p_i \partial p_\ell} - p_\ell \frac{\partial^2 P_\zeta(p)}{\partial p_i \partial p_j} \right] + \mathcal{O}(q^2). \quad (4.14)$$

¹⁴The analyticity properties of $P_\zeta(p)P_\zeta(p+q)\Gamma$ and Γ are obviously the same.

¹⁵The locality assumption was also implicit in [8], for otherwise the effective action would be ill-defined at zero momentum. Indeed, in their approach one obtains

$$\int d^3k \delta^{(3)}(\vec{k}) \frac{\delta\Gamma[\zeta]}{\delta\zeta_{\vec{k}}} = - \int d^3k \zeta_{\vec{k}} \vec{k} \cdot \frac{\partial}{\partial \vec{k}} \frac{\delta\Gamma[\zeta]}{\delta\zeta_{-\vec{k}}}, \quad (4.13)$$

which can be used to derive dilation consistency relation, provided of course that the integrals converge.

Substituting this into the first of (4.11) we obtain

$$\begin{aligned} \frac{\langle \zeta_{\vec{q}} \zeta_{\vec{p}} \zeta_{-\vec{q}-\vec{p}} \rangle'}{P_\zeta(q)} &= - \left(3 + p_k \frac{\partial}{\partial p_k} \right) P_\zeta(p) \\ &\quad - \frac{1}{2} q_k \left(6 \frac{\partial}{\partial p_k} - p_k \frac{\partial^2}{\partial p_a \partial p_a} + 2 p_a \frac{\partial^2}{\partial p_a \partial p_k} \right) P_\zeta(p) + \mathcal{O}(q^2). \end{aligned} \quad (4.15)$$

The first and second lines match respectively the dilation and SCT consistency relations [5]. Similarly, the second of (4.11) gives

$$\begin{aligned} \frac{\langle \gamma_{\vec{q}}^{ij} \zeta_{\vec{p}} \zeta_{-\vec{q}-\vec{p}} \rangle'}{P_\gamma(q)} &= - \frac{1}{2} \hat{P}^{ijkl}(\hat{q}) p_k \frac{\partial}{\partial p_\ell} P_\zeta(p) \\ &\quad + \frac{1}{4} \hat{P}^{ijkl}(\hat{q}) q_m \left(p_m \frac{\partial^2}{\partial p_k \partial p_\ell} - 2 p_k \frac{\partial^2}{\partial p_\ell \partial p_m} \right) P_\zeta(p) + \mathcal{O}(q^2). \end{aligned} \quad (4.16)$$

The first and second lines correctly reproduce the anisotropic scaling [1] and linear gradient tensor consistency relations [5], respectively.

4.3 Higher-order consistency relations

At order q^n , $n \geq 2$, the soft correlation functions are only partially constrained by lower-point functions. The novel consistency relations with two hard scalar modes take the form [9]¹⁶

$$\begin{aligned} \lim_{\vec{q} \rightarrow 0} P_{i\ell_0 \dots \ell_n j m_0 \dots m_n}(\hat{q}) \frac{\partial^n}{\partial q_{m_1} \dots \partial q_{m_n}} &\left(\frac{1}{P_\gamma(q)} \langle \gamma_{\vec{q}}^{jm_0} \zeta_{\vec{p}} \zeta_{-\vec{q}-\vec{p}} \rangle' + \frac{\delta^{jm_0}}{3 P_\zeta(q)} \langle \zeta_{\vec{q}} \zeta_{\vec{p}} \zeta_{-\vec{q}-\vec{p}} \rangle' \right) \\ &= - P_{i\ell_0 \dots \ell_n j m_0 \dots m_n}(\hat{q}) \left(\delta^{jm_0} \frac{\partial^n}{\partial p_{m_1} \dots \partial p_{m_n}} + \frac{p^j}{n+1} \frac{\partial^{n+1}}{\partial p_{m_0} \dots \partial p_{m_n}} \right) P_\zeta(p). \end{aligned} \quad (4.17)$$

The component operator $P_{i\ell_0 \dots \ell_n j m_0 \dots m_n}(\hat{q})$ has the following properties:

1. It is symmetric in the (ℓ_0, \dots, ℓ_n) indices and in the $(m_0 \dots m_n)$ indices.
2. It is symmetric under the interchange of sets of indices:

$$P_{i\ell_0 \dots \ell_n j m_0 \dots m_n}(\hat{q}) = P_{j m_0 \dots m_n i \ell_0 \dots \ell_n}(\hat{q}). \quad (4.18)$$

3. It obeys the trace condition:

$$P_{i\ell\ell\ell_2 \dots \ell_n j m_0 \dots m_n}(\hat{q}) = -\frac{1}{3} P_{\ell i \ell_2 \dots \ell_n j m_0 \dots m_n}(\hat{q}). \quad (4.19)$$

4. It satisfies the transverse condition:

$$\hat{q}^i \left(P_{i\ell_0 \ell_1 \dots \ell_n j m_0 \dots m_n}(\hat{q}) + P_{\ell_0 i \ell_1 \dots \ell_n j m_0 \dots m_n}(\hat{q}) - \frac{2}{3} \delta_{i\ell_0} P_{\ell\ell\ell_1 \dots \ell_n j m_0 \dots m_n}(\hat{q}) \right) = 0. \quad (4.20)$$

¹⁶The analogue of (4.17) in [9] includes higher-order corrections in γ_{ij} . These do not appear here because we are working at tree-level.

See [9] for a systematic construction and explicit expressions of this operator for the first few values of n .

We are now in the position to show how the consistency relations (4.17) follow from our approach. Substituting (4.5) and (4.10), the identity (4.7) implies

$$\begin{aligned} \frac{\langle \gamma_{\vec{q}}^{jm_0} \zeta_{\vec{p}} \zeta_{-\vec{q}-\vec{p}} \rangle'}{P_\gamma(q)} + \frac{\delta^{jm_0}}{3} \frac{\langle \zeta_{\vec{q}} \zeta_{\vec{p}} \zeta_{-\vec{q}-\vec{p}} \rangle'}{P_\zeta(q)} &= \frac{1}{2} \left(\hat{P}^{jm_0 k \ell}(\hat{q}) + \frac{2}{3} \delta^{jm_0} \delta^{k \ell} \right) K_{k \ell} \\ &+ \frac{1}{2} \left(\hat{P}^{jm_0 k \ell}(\hat{q}) + \frac{2}{3} \delta^{jm_0} \delta^{k \ell}(\hat{q}) \right) \epsilon_{k c m} \epsilon_{\ell d n} q^c q^d \left(a(p, q) \delta^{m n} + b(p, q) p^m p^n \right). \end{aligned} \quad (4.21)$$

We are then instructed to differentiate this expression n times with respect to q , and project the result using $P_{i \ell_0 \dots \ell_n j m_0 \dots m_n}(\hat{q})$. In doing so, we use the following identities, which, as shown in appendix C, follow from the properties of the projector:

$$\begin{aligned} P_{i \ell_0 \dots \ell_n j m_0 \dots m_n}(\hat{q}) \left(\hat{P}^{jm_0 k \ell}(\hat{q}) + \frac{2}{3} \delta^{jm_0} \delta^{k \ell} \right) &= 2 P_{i \ell_0 \dots \ell_n (k \ell) m_1 \dots m_n}(\hat{q}); \\ P_{i \ell_0 \dots \ell_n j m_0 \dots m_n}(\hat{q}) \frac{\partial^n \hat{P}^{jm_0 k \ell}(\hat{q})}{\partial q_{m_1} \dots \partial q_{m_n}} &= 0; \\ P_{i \ell_0 \dots \ell_n j m_0 \dots m_n}(\hat{q}) \frac{\partial^n}{\partial q_{m_1} \dots \partial q_{m_n}} \left(\epsilon_{c m}^j \epsilon_{d n}^{m_0} q^c q^d \left(a(p, q) \delta^{m n} + b(p, q) p^m p^n \right) \right) &= 0 \end{aligned} \quad (4.22)$$

It follows that the model-dependent contributions to the identity, encoded in the last line of (4.21), are completely projected out of the consistency relations, as desired. Moreover, it also follows that all q -derivatives go through and hit the K term on the right-hand side of (4.21). The result is

$$\begin{aligned} \lim_{\vec{q} \rightarrow 0} P_{i \ell_0 \dots \ell_n j m_0 \dots m_n}(\hat{q}) \frac{\partial^n}{\partial q_{m_1} \dots \partial q_{m_n}} \left(\frac{1}{P_\gamma(q)} \langle \gamma_{\vec{q}}^{jm_0} \zeta_{\vec{p}} \zeta_{-\vec{q}-\vec{p}} \rangle' + \frac{\delta^{jm_0}}{3 P_\zeta(q)} \langle \zeta_{\vec{q}} \zeta_{\vec{p}} \zeta_{-\vec{q}-\vec{p}} \rangle' \right) \\ = - \lim_{\vec{q} \rightarrow 0} P_{i \ell_0 \dots \ell_n j m_0 \dots m_n}(\hat{q}) \frac{\partial^n}{\partial q_{m_1} \dots \partial q_{m_n}} K_{j m_0}. \end{aligned} \quad (4.23)$$

Finally, using the rotational invariance of the power spectrum, it is straightforward to show that

$$P_{i \ell_0 \dots \ell_n j m_0 \dots m_n}(\hat{q}) \left(p^{m_0} \frac{\partial^{n+1}}{\partial p_j \partial p_{m_1} \dots \partial p_{m_n}} - p^{m_1} \frac{\partial^{n+1}}{\partial p_j \partial p_{m_0} \partial p_{m_2} \dots \partial p_{m_n}} \right) P_\zeta(p) = 0. \quad (4.24)$$

It follows that the last line in the expression (4.8) for K_{ij} projects out. Using this fact, (4.23) clearly reduces to (4.17), as claimed.

The fixed-time path integral method [8] used above, while elegant and transparent, has certain limitations. Although the generating functional W derived from (3.3) determines correctly the various correlation functions, the vertex functional Γ obtained through its Legendre transform is not related straightforwardly to the actual, four-dimensional effective action.¹⁷ On the other hand, the analyticity assumption made in section 4.1, which was critical in deriving the consistency relations, is only well-motivated for the four-dimensional effective action, while the momentum dependence of (the fixed-time) Γ is *a priori* unknown.

¹⁷In practice, the various contributions to Γ are obtained by calculating in-in correlation functions from contact diagrams, and dividing by the power spectrum for each external leg.

To avoid any guesswork, below we will repeat the calculation for the usual time-dependent in-in path integral. In that case the vertex functional encodes effective interaction vertices, which could be read off the quantum action. To leading order in \hbar (i.e., at tree level), Γ simply encodes the interaction vertices of the classical action we started with, i.e. GR + inflaton. In this limit the analyticity assumptions about Γ directly correspond to assumptions about the locality of the Lagrangian of the theory, and hence the arguments of section 4.1 about single-field and constant growing modes are well-motivated.

5 Time-dependent path integral formalism

In this section we show how the general consistency relations follow from spatial diffeomorphism invariance of the conventional, time-dependent path integral formalism. The starting point is the following four-dimensional path integral

$$Z[T, J] = \int \mathcal{D}h_{ij} \mathcal{D}\varphi e^{iS[h, \varphi] + i \int d^4x (h_{ij} T^{ij} + \varphi J)}. \quad (5.1)$$

(Note that we omit the gauge condition, it will be imposed when necessary.) As before, we assume that the lapse function and shift vectors have been integrated out. An important remark is in order here. Whether the above path-integral describes the generator of in-out or in-in diagrams is determined by the time contour, the integration along which determines the action and the source term. For in-out diagrams, the time contour stretches along the real axis $(-\infty, +\infty)$. For in-in diagrams, it lies on the complex plane $(-\infty + i\epsilon, t) \cup (t, -\infty - i\epsilon)$. This approach, followed here, is equivalent to doubling of the fields. The literature on this subject is vast, e.g., see [50] and references therein.

The choice of ζ gauge breaks the time reparametrization symmetry explicitly. The symmetries of interest are therefore spatial diffeomorphisms. Since we are now considering the four-dimensional path integral, in all generality we allow for time-dependent spatial diffeomorphisms $\xi^j(\vec{x}, t)$. Demanding that $Z[T, J]$ be invariant under (3.4) and following similar steps as in section 3, we obtain the Slavnov-Taylor identity:

$$2\partial_j \left(\frac{1}{6} \delta_{ij} \frac{\delta\Gamma}{\delta\zeta} + \frac{\delta\Gamma}{\delta\gamma_{ij}} \right) = \partial_i \zeta \frac{\delta\Gamma}{\delta\zeta} + \dots, \quad (5.2)$$

where we have omitted the gauge-fixing contribution for simplicity. Although this is superficially identical to (3.15) in the fixed-time approach, an important distinction is that Γ now represents the effective action, rather than a quantity defined in terms of the equal-time Green's functions. Varying this identity a number of times with respect to ζ and γ leads to various consistency relations among the vertices of the theory.

As in section 4, we illustrate this with the simplest case of two hard scalar modes coupled to a soft ζ or γ mode. Varying (5.2) with respect to $\zeta(\vec{x}_1, \tau_1)$ and $\zeta(\vec{x}_2, \tau_2)$, and going to momentum space for the spatial dimensions, we obtain the master identity

$$q^j \tilde{\Gamma}_{ij}(\vec{q}, \tau; \vec{p}, \tau_1; -\vec{q} - \vec{p}, \tau_2) = -\delta(\tau - \tau_1) p_i \Gamma_\zeta(|\vec{q} + \vec{p}|; \tau, \tau_2) + \delta(\tau - \tau_2) (q_i + p_i) \Gamma_\zeta(p; \tau, \tau_1), \quad (5.3)$$

where $\tilde{\Gamma}_{ij} \equiv \frac{1}{3} \delta_{ij} \Gamma^{\zeta\zeta\zeta} + 2\Gamma_{ij}^{\gamma\zeta\zeta}$. This can be translated to a statement about correlation

functions using

$$\begin{aligned}\delta(t_1 - t_2) &= \int d\tau P_\zeta(p; t_1, \tau) \Gamma_\zeta(p; t_2, \tau); \\ \langle \zeta_{\vec{q}}(t) \zeta_{\vec{p}}(t_1) \zeta_{-\vec{q}-\vec{p}}(t_2) \rangle' &= - \int d\tau d\tau_1 d\tau_2 P_\zeta(q; \tau, t) P_\zeta(p; \tau_1, t_1) P_\zeta(|\vec{q} + \vec{p}|; \tau_2, t_2) \\ &\quad \times \Gamma^{\zeta\zeta\zeta}(\vec{q}, \tau; \vec{p}, \tau_1; -\vec{q} - \vec{p}, \tau_2),\end{aligned}\tag{5.4}$$

and similarly for tensors. These relations tell us that we should contract (5.3) with two power spectra to obtain:

$$\begin{aligned}q^j \int d\tau_1 d\tau_2 P_\zeta(p; t, \tau_1) P_\zeta(|\vec{q} + \vec{p}|; t, \tau_2) \tilde{\Gamma}_{ij}(\vec{q}, \tau; \vec{p}, \tau_1; -\vec{q} - \vec{p}, \tau_2) \\ = \delta(\tau - t) \left((q_i + p_i) P_\zeta(|\vec{q} + \vec{p}|, \tau) - p_i P_\zeta(p, \tau) \right).\end{aligned}\tag{5.5}$$

As before, this can be solved as a Taylor series around $q = 0$:

$$\int d\tau_1 d\tau_2 P_\zeta(p; t, \tau_1) P_\zeta(|\vec{q} + \vec{p}|; t, \tau_2) \tilde{\Gamma}_{ij}(\vec{q}, \tau; \vec{p}, \tau_1; -\vec{q} - \vec{p}, \tau_2) = \delta(\tau - t) K_{ij} + A_{ij}(p, q, \tau, t),\tag{5.6}$$

where K_{ij} is given by (4.8), with $P_\zeta(p)$ understood as $P_\zeta(p, t)$, and A_{ij} again denotes an arbitrary, symmetric and transverse tensor. (Note that, unlike in section 4, A_{ij} can now depend on two times.) The scalar and tensor vertices are then isolated by taking the trace and traceless parts of (5.6). To extract correlation functions, we multiply the results by the appropriate (unequal-time) power spectra $P(q, t, \tau)$, and integrate over τ :

$$\begin{aligned}\frac{\langle \zeta_{\vec{q}} \zeta_{\vec{p}} \zeta_{-\vec{q}-\vec{p}} \rangle'}{P_\zeta(q)} &= K + \int d\tau \frac{P_\zeta(q, \tau, t)}{P_\zeta(q, t)} A(p, q, \tau, t); \\ \frac{\langle \gamma_{\vec{q}}^{ij} \zeta_{\vec{p}} \zeta_{-\vec{q}-\vec{p}} \rangle'}{P_\gamma(q)} &= \frac{1}{2} \hat{P}^{ijkl}(\hat{q}) \left(K_{kl} + \int d\tau \frac{P_\zeta(q, \tau, t)}{P_\zeta(q, t)} A_{kl}(p, q, \tau, t) \right).\end{aligned}\tag{5.7}$$

To derive consistency relations, recall that we had to assume in section 4.1 that the physical term A_{ij} started at order q^2 , which was motivated by locality. A subtlety, already mentioned at the end of section 4, is that the 3d vertex functional Γ^{3d} considered in the fixed-time path integral formalism is of course not the same as the 4d vertex functional Γ^{4d} of this section. In particular, their analyticity properties may in principle differ. The meaning of locality for Γ^{4d} is clear — it represents the 4d effective action, which at tree-level reduces to the action we started with (so-called ‘fundamental action’). Its locality is guaranteed by the locality of the starting-point Lagrangian. Meanwhile, the 3d vertex functional is given by

$$\Gamma^{3d}(\vec{q}, \vec{p}, -\vec{p} - \vec{q}) \sim \int d\tau d\tau_1 d\tau_2 \frac{P_\zeta(q, t, \tau)}{P_\zeta(q, t)} \frac{P_\zeta(p, t, \tau_1)}{P_\zeta(p, t)} \frac{P_\zeta(|\vec{p} + \vec{q}|, t, \tau_2)}{P_\zeta(|\vec{p} + \vec{q}|, t)} \Gamma^{4d}(\vec{q}, \tau; \vec{p}, \tau_1; -\vec{p} - \vec{q}, \tau_2).\tag{5.8}$$

We see that the locality of Γ^{3d} , on the other hand, follows from that of Γ^{4d} *provided* that the ratio of power spectra $P_\zeta(q, \tau, t)/P_\zeta(q, t)$ is analytic in q . This will be the case if mode functions have constant growing-mode solutions. This additional assumption was implicitly made in section 4. It also implies, incidentally, if A_{ij} starts at order q^2 , then the time integrals in (5.7) will also start at order q^2 . The rest of the derivation proceeds as in section 4, and we recover the consistency relations to all orders in q .

6 Conclusion

In this paper, we have shown that the infinite network of consistency relations for adiabatic modes, of which Maldacena’s relation is the simplest, all follow from a single master identity resulting from the Slavnov-Taylor identity for spatial diffeomorphisms. The master identity is cast in terms of the vertex functional and holds for any momenta. By varying this identity a number of times with respect to the fields, one can obtain consistency relations for the various correlation functions. We have illustrated this for the simplest case of two hard scalar modes coupled to a soft scalar or tensor mode.

One of the key insights of this derivation is that it makes precise the assumption underlying the consistency relations, namely the locality of the effective action in the $q \rightarrow 0$ limit. For the simplest inflationary models, this is equivalent to the standard assumption that mode functions tend to a constant at late times. For more exotic models, in which modes do not “freeze” in the usual sense, locality offers an unambiguous criterion.

The general formalism described here can be applied more broadly to a host other contexts. It should be straightforward to generalize the derivation to include additional scalar fields. As is well known, consistency relations can be violated in the multi-field context, and it would be interesting to see how this shows up in our formalism. Other interesting applications include the path integral derivation of consistency relations for the large scale structure [26–29], the study of modified initial states [30–38], and higher soft limits [39].

Acknowledgments

We would like to thank Yi-Zen Chu, Paolo Creminelli, Kurt Hinterbichler, Lam Hui, Austin Joyce, Guilherme Pimentel, Marko Simonovic and Junpu Wang for useful discussions. L.B. is supported by funds provided by the University of Pennsylvania. J.K. is supported in part by NSF CAREER Award PHY-1145525. While this paper was in its final stages, we became aware of [51], which has some overlap with the results presented here.

A Projection operators for electrodynamics

In this appendix, we show that the projectors $P_{\mu\ell_1\dots\ell_n\nu m_1\dots m_n}$ we have defined for QED (see (2.17) and (2.18)) are sufficient to project out the model-dependent contribution C_μ from (2.16).

First, to ensure that the $q \rightarrow 0$ limit in (2.16) is well defined, we need the following identity to hold

$$P_{\mu\ell_1\dots\ell_n\nu m_1\dots m_n} \frac{\partial^n P^{\nu\alpha}(\hat{q})}{\partial q_{m_1} \dots \partial q_{m_n}} = 0, \quad (\text{A.1})$$

for otherwise the derivatives of $P^{\nu\alpha}(\hat{q})$ would yield singular terms as $q \rightarrow 0$. Equation (A.1) is satisfied, thanks to the properties (2.17) and (2.18) of the projector: the derivatives of $P^{\nu\alpha}$ either trace $P_{\mu\ell_1\dots\ell_n\nu m_1\dots m_n}$ or project it on q^m .

The contribution of C_μ -dependent terms to the consistency relation is of the form:

$$\begin{aligned} & P_{\gamma\ell_1\dots\ell_n\nu m_1\dots m_n} \frac{\partial^n}{\partial q_{m_1} \dots \partial q_{m_n}} (P^{\nu\mu}[(q^2\eta_{\mu\alpha} - q_\mu q_\alpha)v^\alpha + q^\alpha M_{\mu\alpha}]) \\ &= P_{\gamma\ell_1\dots\ell_n\nu m_1\dots m_n} \frac{\partial^n}{\partial q_{m_1} \dots \partial q_{m_n}} (q^2 P_{\nu\alpha} v^\alpha + q^\alpha M_{\nu\alpha}), \end{aligned} \quad (\text{A.2})$$

where in the last step we have used the transversality of $P_{\mu\nu}$ and anti-symmetry of $M_{\mu\nu}$. Using (A.1), and noting that v^α and $M_{\mu\nu}$ are both regular in $q \rightarrow 0$ limit, it follows that the q^2 and q^α factors in (4.10) must be necessarily differentiated. However, differentiating these factors either results in tracing $P_{\gamma\ell_1\ldots\ell_n\nu m_1\ldots m_n}$, which vanishes by (2.17), or contracting it with $M_{\mu\nu}$, which vanishes by symmetry. This shows that the properties of $P_{\mu\ell_1\ldots\ell_n\nu m_1\ldots m_n}$ are sufficient to project out the model-dependent contributions, as claimed.

B Gauge-fixing term for gravity

In this appendix we will justify that, at the tree-level, it is legitimate to neglect the gauge-fixing term, as claimed in section 3. In order to keep the consideration simple, we will be schematic and omit indices. Focusing on the metric degree of freedom h , the generating functional is

$$Z[T] = \int \mathcal{D}h \, e^{iS[h] + f(\frac{1}{2\alpha}(\partial h)^2 + hT)}, \quad (\text{B.1})$$

where T is an external current, and α is a gauge fixing parameter. We assume that the action, as well as the measure, are invariant under the following schematic diffeomorphism:

$$h \rightarrow h + \partial\xi + \xi\partial h. \quad (\text{B.2})$$

The invariance of B.1 under this transformation leads to the following functional differential equation for the generating functional:

$$\left(\frac{1}{\alpha} \partial^3 \frac{\delta}{\delta T(x)} - \partial T(x) + T(x) \partial \frac{\delta}{\delta T(x)} \right) Z[T] = \frac{1}{\alpha} \int \mathcal{D}h e^{iS[h] + f(\frac{1}{2\alpha}(\partial h)^2 + hT)} \partial^2 h(x) \partial h(x). \quad (\text{B.3})$$

The right hand side is the expectation value of the fields evaluated at the same point. In order to extract the desired relations, we first set $Z = \exp(iW)$ and introduce the effective action $\Gamma[h]$ as a Legendre transform of W . The resulting equation for Γ is

$$\frac{1}{\alpha} \partial^3 h(x) - \partial \frac{\delta \Gamma}{\delta h(x)} + \partial h(x) \frac{\delta \Gamma}{\delta h(x)} = e^{-iW} \frac{1}{\alpha} \int \mathcal{D}h e^{iS[h] + f(\frac{1}{2\alpha}(\partial h)^2 + hT)} \partial^2 h(x) \partial h(x). \quad (\text{B.4})$$

The first term on the left-hand side is analogous to the first term of (2.6) — since we must differentiate this equation at least twice with respect to h , this term will not contribute the final identities (similarly to QED). The gauge-fixing contribution on the right-hand side, on the other hand, is not removable. It is divergent and requires regularization. This is one of the complications associated with non-Abelian gauge theories, compared to Abelian ones. Fortunately, this troublesome term can be ignored at tree level — it corresponds to a contribution to the vertex functional where fields are evaluated at the same point, and hence is of loop order.

To summarize, at the tree level the vertex functional satisfies

$$\frac{1}{\alpha} \partial^3 h(x) - \partial \frac{\delta \Gamma}{\delta h(x)} + \partial h(x) \frac{\delta \Gamma}{\delta h(x)} = 0. \quad (\text{B.5})$$

As already mentioned the first term does not contribute to consistency relations. The equation given above is simply a statement about the gauge invariance of the action.¹⁸ In other words, at tree-level there is no need to fix the gauge in vertices. All gauge redundancies are taken care of by the gauge-fixed propagators upon contraction with vertices.

¹⁸At tree-level, Γ coincides with the action S , supplemented by the gauge-fixing term.

C Properties of projectors

In this appendix we derive identities for $P_{i\ell_0\dots\ell_njm_0\dots m_n}$ that are useful in deriving consistency relations. Using the properties listed in section 4.3 and the explicit form of $\hat{P}_{jm_0k\ell}$ from (4.6), it is straightforward to show that

$$P_{i\ell_0\dots\ell_njm_0\dots m_n}(\hat{q})\hat{P}_{jm_0k\ell}(\hat{q}) = P_{i\ell_0\dots\ell_nk\ell\dots m_n}(\hat{q}) + P_{i\ell_0\dots\ell_n\ell k\dots m_n}(\hat{q}) - \frac{2}{3}\delta_{k\ell}P_{i\ell_0\dots\ell_njjm_1\dots m_n}(\hat{q}). \quad (\text{C.1})$$

Note that the structure on the right-hand side is such that it vanishes when hit by q_k , which follows from (4.20). By tracing (C.1) we discover another important property of projectors

$$P_{i\ell_0\dots\ell_njm_0m_1\dots m_n}(\hat{q})\hat{P}_{jm_0m_1\ell}(\hat{q}) = 0. \quad (\text{C.2})$$

In order to obtain the identities involving derivatives of $\hat{P}_{jm_0k\ell}$, we will need the fact that its first derivative can be written as

$$\frac{\partial \hat{P}_{jm_0k\ell}(\hat{q})}{\partial q_{m_1}} = -\frac{1}{q^2} \left(q_j \hat{P}_{k\ell m_0 m_1}(\hat{q}) + q_k \hat{P}_{j m_0 \ell m_1}(\hat{q}) + q_\ell \hat{P}_{j m_0 k m_1}(\hat{q}) + q_{m_0} \hat{P}_{k\ell j m_1}(\hat{q}) \right). \quad (\text{C.3})$$

Contracting this with $P_{i\ell_0\dots\ell_njm_0\dots m_n}$ and using (C.1), we get

$$P_{i\ell_0\dots\ell_njm_0\dots m_n}(\hat{q}) \frac{\partial \hat{P}_{jm_0k\ell}(\hat{q})}{\partial q_{m_1}} = 0. \quad (\text{C.4})$$

Having obtained these basic properties of the projector, we proceed by method of strong induction to show that

$$P_{i\ell_0\dots\ell_njm_0\dots m_n}(\hat{q}) \frac{\partial^n \hat{P}_{jm_0k\ell}(\hat{q})}{\partial q_{m_1} \dots \partial q_{m_n}} = 0; \quad \forall n > 1, \quad (\text{C.5})$$

assuming

$$P_{i\ell_0\dots\ell_njm_0\dots m_{n-i}} \frac{\partial^{n-i} \hat{P}_{jm_0k\ell}(\hat{q})}{\partial q_{m_1} \dots \partial q_{m_{n-i}}} = 0; \quad 1 \leq i < n. \quad (\text{C.6})$$

Taking into account (C.3), we have

$$\begin{aligned} P_{i\ell_0\dots\ell_njm_0\dots m_n} \frac{\partial^n \hat{P}_{jm_0k\ell}(\hat{q})}{\partial q_{m_1} \dots \partial q_{m_n}} &= P_{i\ell_0\dots\ell_njm_0\dots m_n} \\ &\times \frac{\partial^{n-1}}{\partial q_{m_2} \dots \partial q_{m_n}} \left(-\frac{1}{q^2} \left[q_j \hat{P}_{k\ell m_0 m_1} + q_k \hat{P}_{j m_0 \ell m_1} + q_\ell \hat{P}_{j m_0 k m_1} + q_{m_0} \hat{P}_{k\ell j m_1} \right] \right). \end{aligned} \quad (\text{C.7})$$

Upon performing all the differentiations on the right-hand side, we will obtain terms with differentiated $\hat{P}_{jm_0k\ell}$ as well as with undifferentiated ones. According to the properties of the projector given in section 4.3, along with (C.6), the only nonzero terms among those involving derivatives of $\hat{P}_{jm_0k\ell}$ come from differentiating the first term on the right-hand side of (C.7). Furthermore, among these there are terms with differentiated q_j . However, the differentiation of q_j gives us a factor of δ_{jm_2} ; as a result, the trace property of $P_{i\ell_0\dots\ell_njm_0\dots m_n}$ becomes applicable and terms under the consideration are nullified by means of the assumption (C.6).

In other words, the only terms with derivatives of \hat{P}_{jm_0kl} contributing to (C.7) are of the form¹⁹

$$P_{i\ell_0\dots\ell_njm_0\dots m_n} \frac{\partial^n \hat{P}_{jm_0kl}(\hat{q})}{\partial q_{m_1} \dots \partial q_{m_n}} \supset P_{i\ell_0\dots\ell_njm_0\dots m_n} q_j \times \sum_{a>0} \frac{\partial^{n-a-1}}{\partial q_{m_{a+2}} \dots \partial q_{m_{n-1}}} \left(\frac{-1}{q^2} \right) \times \frac{\partial^a \hat{P}_{k\ell m_0 m_1}(\hat{q})}{\partial q_{m_2} \dots \partial q_{m_{a+1}}}. \quad (\text{C.8})$$

Using the transversality property $P_{i\ell_0\dots\ell_njm_0\dots m_n} q_j = -P_{i\ell_0\dots\ell_n m_0 j \dots m_n} q_j + \frac{2}{3} q_{m_0} P_{i\ell_0\dots\ell_n jj \dots m_n}$, (C.8) reduces to²⁰

$$P_{i\ell_0\dots\ell_njm_0\dots m_n} \frac{\partial^n \hat{P}_{jm_0kl}(\hat{q})}{\partial q_{m_1} \dots \partial q_{m_n}} \supset \frac{2}{3} \sum_{a>0} \frac{\partial^{n-a-1}}{\partial q_{m_{a+2}} \dots \partial q_{m_{n-1}}} \left(\frac{-1}{q^2} \right) \times P_{i\ell_0\dots\ell_n jj m_1 \dots m_n} q_{m_0} \frac{\partial^a \hat{P}_{k\ell m_0 m_1}(\hat{q})}{\partial q_{m_2} \dots \partial q_{m_{a+1}}}. \quad (\text{C.9})$$

Now, using the expression for a th derivative of the identity $q_{m_0} \hat{P}_{k\ell m_0 m_1} = 0$, it is easy to see that the only surviving term in the sum will be the one with $a = 1$. Furthermore, there will be $n - 1$ of those terms in (C.7). Hence, the only term involving the derivative \hat{P}_{jm_0kl} , using $q_{m_0} \partial_{q_{m_2}} \hat{P}_{k\ell m_0 m_1} = -\hat{P}_{k\ell m_2 m_1}$, reduces to

$$P_{i\ell_0\dots\ell_njm_0\dots m_n} \frac{\partial^n \hat{P}_{jm_0kl}(\hat{q})}{\partial q_{m_1} \dots \partial q_{m_n}} \supset -\frac{2}{3} (n-1) \frac{\partial^{n-2}}{\partial q_{m_3} \dots \partial q_{m_{n-1}}} \left(\frac{-1}{q^2} \right) P_{i\ell_0\dots\ell_n jj m_1 \dots m_n} \hat{P}_{k\ell m_1 m_2}(\hat{q}). \quad (\text{C.10})$$

The rest of the terms in (C.7) are the ones with no derivative acting on \hat{P}_{jm_0kl} to begin with. The only non-vanishing ones are with one derivative acting on q_i

$$P_{i\ell_0\dots\ell_njm_0\dots m_n} \frac{\partial^n \hat{P}_{jm_0kl}(\hat{q})}{\partial q_{m_1} \dots \partial q_{m_n}} \supset (n-1) \frac{\partial^{n-2}}{\partial q_{m_3} \dots \partial q_{m_{n-1}}} \left(\frac{-1}{q^2} \right) \times P_{i\ell_0\dots\ell_njm_0\dots m_n} \left[\delta_{jm_2} \hat{P}_{k\ell m_0 m_1} + \delta_{km_2} \hat{P}_{jm_0 \ell m_1} + \delta_{\ell m_2} \hat{P}_{jm_0 k m_1} + \delta_{m_0 m_2} \hat{P}_{k\ell j m_1} \right]. \quad (\text{C.11})$$

Combining (C.10) and (C.11), we obtain

$$P_{i\ell_0\dots\ell_njm_0\dots m_n} \frac{\partial^n \hat{P}_{jm_0kl}(\hat{q})}{\partial q_{m_1} \dots \partial q_{m_n}} = (n-1) \frac{\partial^{n-2}}{\partial q_{m_3} \dots \partial q_{m_{n-1}}} \left(\frac{-1}{q^2} \right) \times \left(-\frac{2}{3} P_{i\ell_0\dots\ell_n jj m_1 \dots m_n} \hat{P}_{k\ell m_1 m_2} + P_{i\ell_0\dots\ell_njm_0\dots m_n} \left[\delta_{jm_2} \hat{P}_{k\ell m_0 m_1} + \delta_{m_0 m_2} \hat{P}_{k\ell j m_1} \right] \right), \quad (\text{C.12})$$

where we have used (C.2). The right-hand side vanishes, once we use the trace property of the projector. In other words, the identity (C.5) holds, as we wanted to show.

References

- [1] J.M. Maldacena, *Non-Gaussian features of primordial fluctuations in single field inflationary models*, *JHEP* **05** (2003) 013 [[astro-ph/0210603](#)] [[INSPIRE](#)].

¹⁹Here, every term in the sum implicitly includes an appropriate multiplicity factor, which we omit for simplicity.

²⁰We have used (C.6) here again.

- [2] P. Creminelli and M. Zaldarriaga, *Single field consistency relation for the 3-point function*, *JCAP* **10** (2004) 006 [[astro-ph/0407059](#)] [[INSPIRE](#)].
- [3] C. Cheung, A.L. Fitzpatrick, J. Kaplan and L. Senatore, *On the consistency relation of the 3-point function in single field inflation*, *JCAP* **02** (2008) 021 [[arXiv:0709.0295](#)] [[INSPIRE](#)].
- [4] L. Senatore and M. Zaldarriaga, *A Note on the Consistency Condition of Primordial Fluctuations*, *JCAP* **08** (2012) 001 [[arXiv:1203.6884](#)] [[INSPIRE](#)].
- [5] P. Creminelli, J. Norena and M. Simonovic, *Conformal consistency relations for single-field inflation*, *JCAP* **07** (2012) 052 [[arXiv:1203.4595](#)] [[INSPIRE](#)].
- [6] K. Hinterbichler, L. Hui and J. Khoury, *Conformal Symmetries of Adiabatic Modes in Cosmology*, *JCAP* **08** (2012) 017 [[arXiv:1203.6351](#)] [[INSPIRE](#)].
- [7] V. Assassi, D. Baumann and D. Green, *On Soft Limits of Inflationary Correlation Functions*, *JCAP* **11** (2012) 047 [[arXiv:1204.4207](#)] [[INSPIRE](#)].
- [8] W.D. Goldberger, L. Hui and A. Nicolis, *One-particle-irreducible consistency relations for cosmological perturbations*, *Phys. Rev. D* **87** (2013) 103520 [[arXiv:1303.1193](#)] [[INSPIRE](#)].
- [9] K. Hinterbichler, L. Hui and J. Khoury, *An Infinite Set of Ward Identities for Adiabatic Modes in Cosmology*, [arXiv:1304.5527](#) [[INSPIRE](#)].
- [10] J.M. Bardeen, P.J. Steinhardt and M.S. Turner, *Spontaneous Creation of Almost Scale - Free Density Perturbations in an Inflationary Universe*, *Phys. Rev. D* **28** (1983) 679 [[INSPIRE](#)].
- [11] D. Salopek and J. Bond, *Nonlinear evolution of long wavelength metric fluctuations in inflationary models*, *Phys. Rev. D* **42** (1990) 3936 [[INSPIRE](#)].
- [12] Y.-F. Cai, W. Xue, R. Brandenberger and X. Zhang, *Non-Gaussianity in a Matter Bounce*, *JCAP* **05** (2009) 011 [[arXiv:0903.0631](#)] [[INSPIRE](#)].
- [13] J. Khoury and F. Piazza, *Rapidly-Varying Speed of Sound, Scale Invariance and Non-Gaussian Signatures*, *JCAP* **07** (2009) 026 [[arXiv:0811.3633](#)] [[INSPIRE](#)].
- [14] M.H. Namjoo, H. Firouzjahi and M. Sasaki, *Violation of non-Gaussianity consistency relation in a single field inflationary model*, *Europhys. Lett.* **101** (2013) 39001 [[arXiv:1210.3692](#)] [[INSPIRE](#)].
- [15] X. Chen, H. Firouzjahi, M.H. Namjoo and M. Sasaki, *A Single Field Inflation Model with Large Local Non-Gaussianity*, *Europhys. Lett.* **102** (2013) 59001 [[arXiv:1301.5699](#)] [[INSPIRE](#)].
- [16] X. Chen, H. Firouzjahi, E. Komatsu, M.H. Namjoo and M. Sasaki, *In-in and δN calculations of the bispectrum from non-attractor single-field inflation*, *JCAP* **12** (2013) 039 [[arXiv:1308.5341](#)] [[INSPIRE](#)].
- [17] T. Tanaka and Y. Urakawa, *Dominance of gauge artifact in the consistency relation for the primordial bispectrum*, *JCAP* **05** (2011) 014 [[arXiv:1103.1251](#)] [[INSPIRE](#)].
- [18] E. Pajer, F. Schmidt and M. Zaldarriaga, *The Observed Squeezed Limit of Cosmological Three-Point Functions*, [arXiv:1305.0824](#) [[INSPIRE](#)].
- [19] K. Schalm, G. Shiu and T. van der Aalst, *Consistency condition for inflation from (broken) conformal symmetry*, *JCAP* **03** (2013) 005 [[arXiv:1211.2157](#)] [[INSPIRE](#)].
- [20] A. Bzowski, P. McFadden and K. Skenderis, *Holography for inflation using conformal perturbation theory*, *JHEP* **04** (2013) 047 [[arXiv:1211.4550](#)] [[INSPIRE](#)].
- [21] S. Weinberg, *Adiabatic modes in cosmology*, *Phys. Rev. D* **67** (2003) 123504 [[astro-ph/0302326](#)] [[INSPIRE](#)].
- [22] P. Creminelli, J. Norena, M. Pena and M. Simonovic, *Khronon inflation*, *JCAP* **11** (2012) 032 [[arXiv:1206.1083](#)] [[INSPIRE](#)].

- [23] A. Slavnov, *Ward Identities in Gauge Theories*, *Theor. Math. Phys.* **10** (1972) 99 [*Teor. Mat. Fiz.* **10** (1972) 153] [[INSPIRE](#)].
- [24] J.C. Ward, *An Identity in Quantum Electrodynamics*, *Phys. Rev.* **78** (1950) 182 [[INSPIRE](#)].
- [25] Y. Takahashi, *On the generalized Ward identity*, *Nuovo Cim.* **6** (1957) 371 [[INSPIRE](#)].
- [26] A. Kehagias and A. Riotto, *Symmetries and Consistency Relations in the Large Scale Structure of the Universe*, *Nucl. Phys. B* **873** (2013) 514 [[arXiv:1302.0130](#)] [[INSPIRE](#)].
- [27] M. Peloso and M. Pietroni, *Galilean invariance and the consistency relation for the nonlinear squeezed bispectrum of large scale structure*, *JCAP* **05** (2013) 031 [[arXiv:1302.0223](#)] [[INSPIRE](#)].
- [28] P. Creminelli, J. Norena, M. Simonovic and F. Vernizzi, *Single-Field Consistency Relations of Large Scale Structure*, *JCAP* **12** (2013) 025 [[arXiv:1309.3557](#)] [[INSPIRE](#)].
- [29] B. Horn and L. Hui, *Consistency relations in large scale structure*, to appear.
- [30] R. Holman and A.J. Tolley, *Enhanced Non-Gaussianity from Excited Initial States*, *JCAP* **05** (2008) 001 [[arXiv:0710.1302](#)] [[INSPIRE](#)].
- [31] P.D. Meerburg, J.P. van der Schaar and P.S. Corasaniti, *Signatures of Initial State Modifications on Bispectrum Statistics*, *JCAP* **05** (2009) 018 [[arXiv:0901.4044](#)] [[INSPIRE](#)].
- [32] P.D. Meerburg, J.P. van der Schaar and M.G. Jackson, *Bispectrum signatures of a modified vacuum in single field inflation with a small speed of sound*, *JCAP* **02** (2010) 001 [[arXiv:0910.4986](#)] [[INSPIRE](#)].
- [33] J. Ganc, *Calculating the local-type f_{NL} for slow-roll inflation with a non-vacuum initial state*, *Phys. Rev. D* **84** (2011) 063514 [[arXiv:1104.0244](#)] [[INSPIRE](#)].
- [34] D. Chialva, *Signatures of very high energy physics in the squeezed limit of the bispectrum (violation of Maldacena's condition)*, *JCAP* **10** (2012) 037 [[arXiv:1108.4203](#)] [[INSPIRE](#)].
- [35] N. Agarwal, R. Holman, A.J. Tolley and J. Lin, *Effective field theory and non-Gaussianity from general inflationary states*, *JHEP* **05** (2013) 085 [[arXiv:1212.1172](#)] [[INSPIRE](#)].
- [36] R. Flauger, D. Green and R.A. Porto, *On squeezed limits in single-field inflation. Part I*, *JCAP* **08** (2013) 032 [[arXiv:1303.1430](#)] [[INSPIRE](#)].
- [37] A. Ashoorioon, K. Dimopoulos, M. Sheikh-Jabbari and G. Shiu, *Reconciliation of High Energy Scale Models of Inflation with Planck*, [arXiv:1306.4914](#) [[INSPIRE](#)].
- [38] A. Aravind, D. Lorshbough and S. Paban, *Non-Gaussianity from Excited Initial Inflationary States*, *JHEP* **07** (2013) 076 [[arXiv:1303.1440](#)] [[INSPIRE](#)].
- [39] A. Joyce, J. Khoury and M. Simonovic, *Higher Soft Limits in Cosmology*, to appear.
- [40] V. Rubakov, *Harrison-Zeldovich spectrum from conformal invariance*, *JCAP* **09** (2009) 030 [[arXiv:0906.3693](#)] [[INSPIRE](#)].
- [41] K. Hinterbichler and J. Khoury, *The Pseudo-Conformal Universe: Scale Invariance from Spontaneous Breaking of Conformal Symmetry*, *JCAP* **04** (2012) 023 [[arXiv:1106.1428](#)] [[INSPIRE](#)].
- [42] P. Creminelli, A. Nicolis and E. Trincherini, *Galilean Genesis: An Alternative to inflation*, *JCAP* **11** (2010) 021 [[arXiv:1007.0027](#)] [[INSPIRE](#)].
- [43] K. Hinterbichler, A. Joyce and J. Khoury, *Non-linear Realizations of Conformal Symmetry and Effective Field Theory for the Pseudo-Conformal Universe*, *JCAP* **06** (2012) 043 [[arXiv:1202.6056](#)] [[INSPIRE](#)].
- [44] P. Creminelli, K. Hinterbichler, J. Khoury, A. Nicolis and E. Trincherini, *Subluminal Galilean Genesis*, *JHEP* **02** (2013) 006 [[arXiv:1209.3768](#)] [[INSPIRE](#)].

- [45] K. Hinterbichler, A. Joyce, J. Khoury and G.E. Miller, *DBI Realizations of the Pseudo-Conformal Universe and Galilean Genesis Scenarios*, *JCAP* **12** (2012) 030 [[arXiv:1209.5742](#)] [[INSPIRE](#)].
- [46] K. Hinterbichler, A. Joyce, J. Khoury and G.E. Miller, *DBI Genesis: An Improved Violation of the Null Energy Condition*, *Phys. Rev. Lett.* **110** (2013) 241303 [[arXiv:1212.3607](#)] [[INSPIRE](#)].
- [47] P. Creminelli, A. Joyce, J. Khoury and M. Simonovic, *Consistency Relations for the Conformal Mechanism*, *JCAP* **04** (2013) 020 [[arXiv:1212.3329](#)] [[INSPIRE](#)].
- [48] D. Capper, G. Leibbrandt and M. Ramon Medrano, *Calculation of the graviton selfenergy using dimensional regularization*, *Phys. Rev. D* **8** (1973) 4320 [[INSPIRE](#)].
- [49] D. Capper and M. Medrano, *Gravitational slavnov-ward identities*, *Phys. Rev. D* **9** (1974) 1641 [[INSPIRE](#)].
- [50] J. Berges, *Introduction to nonequilibrium quantum field theory*, *AIP Conf. Proc.* **739** (2005) 3 [[hep-ph/0409233](#)] [[INSPIRE](#)].
- [51] G.L. Pimentel, *Inflationary Consistency Conditions from a Wavefunctional Perspective*, [arXiv:1309.1793](#) [[INSPIRE](#)].